Birational Maps of Del Pezzo Fibrations.

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1. Introduction.

In classical result, it is known that any $\mathbb{P}^1$-bundle over a nonsingular complex curve T can be birationally transformed to a $\mathbb{P}^1$-bundle over T by an elementary transformation. Here, we can ask if it is also possible in 3-fold case. In other words, is it true that any nonsingular del Pezzo fibration over a nonsingular curve can be transformed to another nonsingular del Pezzo fibration? In this question, we can add more condition on del Pezzo fibrations with some kind of analogue from ruled surface cases, that is, we can assume that their fibers are always nonsingular even though this is not true for any nonsingular del Pezzo fibration.

We ask the same question for local cases. Of course, we can birationally transform any $\mathbb{P}^1$-bundle over a germ of nonsingular complex curve (T, o) into another $\mathbb{P}^1$-bundle over (T, o). But, in del Pezzo fibrations over (T, o), something different happens. In this paper, we will show that any del Pezzo fibration of degree $d \leq 4$ with nonsingular special fiber cannot be birationally transformed into another del Pezzo fibration with nonsingular special fiber.

Let $\mathcal{O}$ be a discrete valuation ring such that its residue field k is of characteristic zero. We denote K the quotient field of $\mathcal{O}$. Let X_K be a variety defined over $\operatorname{Spec} K$. A model of X_K is a flat scheme X defined over $\operatorname{Spec}\mathcal{O}$ whose generic fiber is isomorphic to X_K. Fano fibrations are models of nonsingular Fano variety defined over K. In particular, del Pezzo fibrations of degree d are models of nonsingular del Pezzo surfaces of degree d defined over K. Del Pezzo fibrations are studied in [C96] and [K97]. They constructed "standard model" ([C96]) and "semistable model" ([K97]) in each paper.

Now, we state the theorem which we will prove in this paper.

Main Theorem. *Let X and Y be del Pezzo fibrations of degree $d \leq 4$ over* $\operatorname{Spec}\mathcal{O}$. *Suppose that each scheme-theoretic special fiber is nonsingular. Then any birational map between X and Y over* $\operatorname{Spec}\mathcal{O}$ *which is identical over generic fiber is a biregular morphism.*

We should remark here that even though it is hard to find such examples, there are del Pezzo fibrations of degree $d \leq 4$ over $\operatorname{Spec}\mathcal{O}$ with nonsingular special fibers which can be birationally transformed into another del Pezzo fibration over $\operatorname{Spec}\mathcal{O}$ with reduced and irreducible special fiber. But, as in Minimal model program over 3-folds, we have to allow some mild singularities, such as terminal ones, on them. In the end of this paper, we will give such examples.

From now on, we explain standard definitions and notations for this paper. For more detail, we can refer to [K92], [P99b], and [Sh93].

A variety X means an integral scheme of finite type over a fixed field k. A log pair (X, B) is a normal variety X equipped with a $\mathbb{Q}$-Weil divisor B such that $K_X + B$ is $\mathbb{Q}$-Cartier. A log variety is a log pair (X, B) such that B is a subboundary.

The discrepancy of a divisor E over X with respect to a log pair (X, B) will be denoted by $a(E; X, B)$. And we will use the standard abbreviation plt, klt, and lc for purely log terminal, Kawamata log terminal, and log canonical, respectively.

Let (X, B) be an lc pair and D an effective $\mathbb{Q}$-Cartier divisor on X. The log canonical threshold (or lc threshold) of D is the number

$$lct(X, B, D) := \sup\{c \mid (X, B + cD) \ \ is \ \ lc\}.$$

If $B = 0$, then we use $lct(X, D)$ instead of $lct(X, 0, D)$.

Finally, we will use V. V. Shokurov's 1-complement which is a main tool for this paper. Let X be a normal variety and let D be a reduced and irreducible divisor on X. A divisor $K_X + D$, not necessarily log canonical, is 1-complementary if there is an integral Weil divisor D^+ such that $K_X + D^+$ is linearly trivial, $K_X + D^+$ is lc, and $D^+ \geq D$. The divisor $K_X + D^+$ is called a 1-complement of $K_X + D$. This is just special case of n-complements. But, it is enough for this paper. For more detail about complements, we can refer to [P99b], [Sh93], or [Sh97].

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2. Properties of certain birational maps.

Let $\mathcal{O}$ be a discrete valuation ring with local parameter t. The quotient field and residue field of $\mathcal{O}$ are denoted by K and k, respectively. We always assume that the field k is of characteristic zero. We denote $T = \text{Spec } \mathcal{O}$. For a scheme $\pi : Z \longrightarrow T$, its scheme-theoretic special fiber $\pi^*(o)$ is denoted by S_Z, where o is the closed point of T. From now on, a birational map is always assumed to be identical when restricted to the generic fibers.

Let X/T be a $\mathbb{Q}$-factorial Gorenstein model of a nonsingular variety defined over K which satisfies the following three conditions.

- (Special fiber condition)
  The special fiber S_X is a reduced and irreducible variety with nonempty anticanonical linear system. Moreover, log pair (X, S_X) is plt.

- (1-complement condition)
  For any $C \in |-K_{S_X}|$, there exists 1-complement $K_{S_X} + C_X$ of K_{S_X} such that C_X does not contain any center of log canonicity of $K_{S_X} + C$.

- (Surjectivity condition)
  Any 1-complement of K_{S_X} can be extended to a 1-complement of $K_X + S_X$.

With Special fiber condition, we can easily show that X has at worst terminal singularities. Moreover, the special fiber S_X is a variety over k with Gorenstein canonical singularities.

Let $\phi : X \dashrightarrow Y$ be a birational map over T, where X and Y are $\mathbb{Q}$-factorial Gorenstein models of a nonsingular variety defined over K which satisfy above three conditions. Suppose that $\phi : X \dashrightarrow Y$ is not an isomorphism in codimension 1. We fix a resolution of indeterminacy of $\phi : X \dashrightarrow Y$ as follows.

$$\begin{tikzcd} & W \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X \arrow[rr, dashed, "\phi"] \arrow[dr] & & Y \arrow[dl] \\ & T & \end{tikzcd}$$

Let $\widetilde{S_X}$ and $\widetilde{S_Y}$ be proper transformations of S_X and S_Y by f and g, respectively. Since birational map ϕ is not an isomorphism in codimension 1, $\widetilde{S_X}$ is a g-exceptional divisor and $\widetilde{S_Y}$ is f-exceptional.

Lemma 2.1. *Let $K_X + S_X + D_X$ be a 1-complement of $K_X + S_X$. And let $D_Y = \phi_* D_X$. For any prime divisor E over X,*

$$a(E; X, qS_X + D_X) = a(E; Y, \alpha_q S_Y + D_Y),$$

where q is any given number and $\alpha_q = -a(\widetilde{S_Y}; X, qS_X + D_X)$. Moreover, log canonical divisor $K_Y + S_Y + D_Y$ is linearly trivial.

Proof. Suppose that E is a divisor on W. Note that $f_*^{-1} D_X = g_*^{-1} D_Y = D_W$. Then we have

$$K_W + q\widetilde{S_X} + D_W = f^*(K_X + qS_X + D_X) - \alpha_q \widetilde{S_Y} + \sum a_i E_i,$$

and

$$K_W + \alpha_q \widetilde{S_Y} + D_W = g^*(K_Y + \alpha_q S_Y + D_Y) + b\widetilde{S_X} + \sum b_i E_i,$$

where each E_i is f-exceptional and g-exceptional. From them, we get

$$f^*(K_X + qS_X + D_X) - g^*(K_Y + \alpha_q S_Y + D_Y) = (q + b)\widetilde{S_X} + \sum (b_i - a_i) E_i.$$

Since $K_X + qS_X + D_X$ is numerically trivial, we have

$$(q + b)\widetilde{S_X} + \sum (b_i - a_i) E_i \equiv_g 0.$$

By Negativity lemma, $b = -q$ and $b_i = a_i$. This prove the first statement.

Since ϕ is identical on generic fiber, it is clear that D_Y is linearly equivalent to $-K_Y$. Thus, the second statement follows from the fact that S_Y is linearly trivial. Q.E.D.

Lemma 2.2. *There exists 1-complement $K_{S_X} + C_X$ (resp. $K_{S_Y} + C_Y$) of K_{S_X} (resp. K_{S_Y}) does not contain the center of S_Y (resp. S_X) on X (resp. Y).*

Proof. Let $K_Y + S_Y + L_Y$ be a 1-complement of $K_Y + S_Y$. By lemma 2.1, $a(\widetilde{S_Y}; X, S_X + L_Y) \leq -1$, where $L_X = \phi_*^{-1} L_Y$. Clearly, the center of $\widetilde{S_Y}$ on X is contained in $C = L_X|_{S_X}$. By inversion of adjunction, the center of $\widetilde{S_Y}$ is a center of log canonicity singularities of $K_{S_X} + C$. Furthermore, $K_{S_X} + C$ is linearly trivial by lemma 2.1. Therefore, 1-complement condition implies the statement. Q.E.D.

Lemma 2.3. *There is a 1-complement $K_X + S_X + D_X$ (resp. $K_Y + S_Y + H_Y$) of $K_X + S_X$ (resp. $K_Y + S_Y$) such that D_X (resp. H_Y) does not contain the center of S_Y (resp. S_X).*

Proof. It immediately follows from lemma 2.2 and Surjectivity condition. Q.E.D.

From now on, we fix 1-complements $K_X + S_X + D_X$ and $K_Y + S_Y + H_Y$ of $K_X + S_X$ and $K_Y + S_Y$, respectively, which satisfy the condition in lemma 2.3. We will use the notation D_Y, D_W, H_X and H_W for $\phi_* D_X$, $f_*^{-1} D_X$, $\phi_*^{-1} H_Y$ and $g_*^{-1} H_Y$, respectively. Note that $g_*^{-1} D_Y = f_*^{-1} D_X$ and $g_*^{-1} H_Y = f_*^{-1} H_X$.

Now, we define the following condition.

- (Total lc threshold condition)
  The inequality $\tau_X + \tau_Y > 1$ holds, where $\tau_X = \min\{lct(S_X, C) : C \in |-K_{S_X}|\}$ and $\tau_Y = \min\{lct(S_Y, C) : C \in |-K_{S_Y}|\}$.

Theorem 2.4. *Under Total lc threshold condition, birational map ϕ is an isomorphism in codimension 1.*

Proof. Suppose that ϕ is not an isomorphism in codimension 1. We pay attention to the following eight equations;

$$K_W = f^*(K_X) + a\widetilde{S_Y} + \sum a_i E_i, \qquad \widetilde{S_X} = f^*(S_X) - b\widetilde{S_Y} - \sum b_i E_i,$$

$$D_W = f^*(D_X) - \sum c_i E_i, \qquad H_W = f^*(H_X) - e\widetilde{S_Y} - \sum e_i E_i,$$

$$K_W = g^*(K_Y) + n\widetilde{S_X} + \sum n_i E_i, \qquad \widetilde{S_Y} = g^*(S_Y) - m\widetilde{S_X} - \sum m_i E_i,$$

$$D_W = g^*(D_Y) - l\widetilde{S_X} - \sum l_i E_i, \qquad H_W = g^*(H_Y) - \sum r_i E_i.$$

First of all, $b = m = 1$ since S_X and S_Y are reduced and irreducible. Since D_X does not contain the center of $\widetilde{S_Y}$ on X, we have $\operatorname{mult}_{\widetilde{S_Y}} D_X = 0$. For the same reason, we also have $\operatorname{mult}_{\widetilde{S_X}} H_Y = 0$.

By lemma 2.1, we get $n + a - l = a(\widetilde{S_X}; Y, -aS_Y + D_Y) = a(\widetilde{S_X}; X, D_X) = 0$ and $a + n - e = a(\widetilde{S_Y}; X, -nS_X + H_X) = a(\widetilde{S_Y}; Y, H_Y) = 0$, and hence $a + n = l = e$. Since X and Y have at worst terminal singularities, $a + n = l > 0$.

Since $K_Y + S_Y + D_Y$ is linearly trivial by lemma 2.1, $(K_Y + S_Y + D_Y)|_{S_Y} = K_{S_Y} + D_Y|_{S_Y}$ is linearly trivial. Thus, $D_Y|_{S_Y} \in |-K_{S_Y}|$. Consequently, it follows from inversion of adjunction that $K_X + S_X + \tau_X H_X$ is lc. By the same reason, $K_Y + S_Y + \tau_Y D_Y$ is lc.

Now, we have $a(\widetilde{S_Y}; X, S_X + \tau_X H_X) = a - 1 - \tau_X e \geq -1$ and $a(\widetilde{S_X}; Y, S_Y + \tau_Y D_Y) = n - 1 - \tau_Y l \geq -1$. But, $l = a + n \geq \tau_X e + \tau_Y l = (\tau_X + \tau_Y) l > l$ by Total lc threshold condition. Since $l > 0$, this is impossible. Q.E.D.

3. Lc thresholds on nonsingular del Pezzo surfaces.

Nonsingular del Pezzo surfaces were quite fully studied long time ago. Furthermore, we understand singular del Pezzo surfaces very well. For example, [BW79], [D80], [HW81], and [R94] give us rich information. In this section, we will study some classical result on anticanonical linear systems on del Pezzo surfaces with a modern point of view. Strictly speaking, we investigate all possible singular effective anticanonical divisors on nonsingular del Pezzo surfaces. From this investigation, we can get some information on lc thresholds on nonsingular del Pezzo surfaces.

Lemma 3.1. *Let S be a nonsingular del Pezzo surface of degree $d \leq 4$. Then, $K_S + C$ is lc in codimension 1 for any $C \in |-K_S|$.*

Proof. Let $C = \sum_{i=1}^n m_i C_i \in |-K_S|$, where C_i's are distinct integral curves on S and each $m_i \geq 1$.

First, we claim that if C is not irreducible, then each C_i is isomorphic to $\mathbb{P}^1$. Suppose that C_i is not isomorphic to $\mathbb{P}^1$. Then, the self-intersection number of C_i is greater than 0. Because $-K_S$ is ample, C is connected. So, we have

$$2p_a(C_i) - 2 = (C_i + K_S) \cdot C_i = (1 - m_i)C_i^2 - \sum_{i \neq j} m_j C_j \cdot C_i < 0,$$

which is contradiction. Thus, each component is a nonsingular rational curve.

Since $d = C \cdot (-K_S) = \sum_{i=1}^n m_i C_i \cdot (-K_S)$ and $-K_S$ is ample, we have $\sum_{i=1}^n m_i \leq d$.

If $d = 1$, then $n = 1$ and $m_1 = 1$.

If $d = 2$, then we have three possibilities C_1, $C_1 + C_2$, and $2C_1$. But the last case is absurd because the Fano index of S is one.

Suppose $d = 3$. Then possibilities are C_1, $C_1 + C_2$, $C_1 + C_2 + C_3$, $C_1 + 2C_2$, $2C_1$, and $3C_1$. With the Fano index one, we can get rid of the last two cases. For the case of $C = C_1 + 2C_2$, we consider the equation $3 = K_S^2 = (C_1 + 2C_2)^2 = C_1^2 + 4C_1 \cdot C_2 + 4C_2^2$. Since $(C_1 + 2C_2) \cdot (-K_S) = 3$, we have $C_1 \cdot (-K_S) = C_2 \cdot (-K_S) = 1$, and hence $C_1^2 = C_2^2 = -1$. Thus, $C_1 \cdot C_2 = 2$. But, this implies contradiction $-2 = 2p_a(C_1) - 2 = C_1 \cdot (-2C_2) = -4$.

Finally, we suppose that $d = 4$. We have eleven candidates, C_1, $C_1 + C_2$, $C_1 + C_2 + C_3$, $C_1 + C_2 + C_3 + C_4$, $C_1 + 2C_2$, $C_1 + 3C_2$, $C_1 + C_2 + 2C_3$, $2C_1 + 2C_2$, $2C_1$, $3C_1$, and $4C_1$. Again, we can exclude the last four candidates by Fano index. For the case of $C = C_1 + 3C_2$, we consider the equation $4 = K_S^2 = (C_1 + 3C_2)^2 = C_1^2 + 6C_1 \cdot C_2 + 9C_2^2$. As before, we can see $C_1^2 = C_2^2 = -1$. So, we have contradiction $3C_1 \cdot C_2 = 7$. Let's consider the case of $C = C_1 + 2C_2$. Since $(C_1 + 2C_2) \cdot (-K_S) = 4$, $C_1^2 = 0$ and $C_2^2 = -1$. Then, we have $4 = (C_1 + 2C_2)^2 = -4 + 4C_1 \cdot C_2$. But, $-2 = p_a(C_1) - 2 = -2C_1 \cdot C_2$. Finally, we consider $C = C_1 + C_2 + 2C_3$. Then, each C_i is -1-curve. Since $4 = (C_1 + C_2 + 2C_3)^2 = C_1^2 + C_2^2 + 4C_3^2 + 2(C_1 \cdot C_2 + 2C_1 \cdot C_3 + 2C_2 \cdot C_3)$, we have $5 = C_1 \cdot C_2 + 2C_1 \cdot C_3 + 2C_2 \cdot C_3$. But, $-2 = 2p_a(C_1) - 2 = -(C_2 + 2C_3) \cdot C_1$, and hence $3 = 2C_2 \cdot C_3$. But this is impossible. Q.E.D.

Let S be a nonsingular del Pezzo surface with Fano index r. Then, there is an ample integral divisor H, called fundamental class of S, such that $-K_S = rH$. A curve C on S is called a line (resp. conic and cubic) if $C \cdot H = 1$ (resp. 2 and 3).

Proposition 3.2. *Let S be a nonsingular del Pezzo surface of degree $d \leq 4$ and let $C \in |-K_S|$. Suppose that $K_S + C$ is worse than lc.*

1. *If $d = 1$, then C is a cuspidal rational curve.*

2. *If $d = 2$, then C is one of the following;*

   - *$C = C_1 + C_2$, where C_1 and C_2 are lines intersecting tangentially at one point with $C_1 \cdot C_2 = 2$.*
   - *C is a cuspidal rational curve.*

3. *If $d = 3$, then C is one of the following;*

   - *$C = C_1 + C_2 + C_3$, where C_1, C_2, and C_3 are lines intersecting at one point with $C_1 \cdot C_2 = C_1 \cdot C_3 = C_2 \cdot C_3 = 1$.*
   - *$C = C_1 + C_2$, where C_1 and C_2 are a line and a conic intersecting tangentially at one point with $C_1 \cdot C_2 = 2$.*
   - *C is a cuspidal rational curve.*

4. *If $d = 4$, then C is one of the following;*

   - *$C = C_1 + C_2 + C_3$, where C_1 and C_2 are lines, and C_3 is a conic intersecting at one point with $C_1 \cdot C_2 = C_1 \cdot C_3 = C_2 \cdot C_3 = 1$.*
   - *$C = C_1 + C_2$, where C_1 and C_2 are a line and a cubic intersecting tangentially at one point with $C_1 \cdot C_2 = 2$.*
   - *$C = C_1 + C_2$, where C_1 and C_2 are conics intersecting tangentially at one point with $C_1 \cdot C_2 = 2$.*
   - *C is a cuspidal rational curve.*

Proof. Note that if C is irreducible, then arithmetic genus $p_a(C)$ of C is one. If C is not irreducible, then each component is isomorphic to $\mathbb{P}^1$. And we can see the intersection numbers of two different components of C are less than or equal to 2.

We can easily check the cases of degree 1 and 2.

Now, we suppose that $d = 3$. And we suppose that $C = C_1 + C_2 + C_3$. Since $3 = (C_1 + C_2 + C_3) \cdot (-K_S)$, each C_i is a line. From $2 = 2 - 2p_a(C_1) = C_1 \cdot (C_2 + C_3)$ and $3 = C_1^2 + C_2^2 + C_3^2 + 2C_1 \cdot (C_2 + C_3) + 2C_2 \cdot C_3$, we get $C_2 \cdot C_3 = 1$. Similarly, we can get $C_1 \cdot C_2 = C_1 \cdot C_3 = 1$. Since $K_S + C$ is not lc, these three lines intersect each other at one point.

If C has less than 4 components, then we can show our statement with the same method as above.

The only remaining that we have to show is that $K_S + C$ is lc if $d = 4$ and $C = C_1 + C_2 + C_3 + C_4$. Since each C_i is a line, we get

$$4 = C^2 = -4 + 2(C_1 \cdot C_2 + C_1 \cdot C_3 + C_1 \cdot C_4 + C_2 \cdot C_3 + C_2 \cdot C_4 + C_3 \cdot C_4).$$

And, we have $C_1 \cdot (C_2 + C_3 + C_4) = 2 - 2p_a(C_1) = 2$, $C_2 \cdot (C_1 + C_3 + C_4) = 2$, $C_3 \cdot (C_1 + C_2 + C_4) = 2$, and $C_4 \cdot (C_1 + C_2 + C_3) = 2$. With these 5 equations and connectedness of C, we can see that C is a normal crossing divisor. Thus, $K_S + C$ is lc. Q.E.D.

Corollary 3.3. *Let S be a nonsingular del Pezzo surface of degree $d \leq 4$.*

- *If $d = 1$, then $K_S + \frac{5}{6}C$ is lc for any $C \in |-K_S|$.*
- *If $d = 2$, then $K_S + \frac{3}{4}C$ is lc for any $C \in |-K_S|$.*
- *If $d = 3$ or 4, then $K_S + \frac{2}{3}C$ is lc for any $C \in |-K_S|$.*

Proof. If C is three nonsingular curves intersecting each other at single point transversally, then $lct(X, C) = \frac{2}{3}$. If $C = C_1 + C_2$ where C_i's are nonsingular curves intersecting tangentially with $C_1 \cdot C_2 = 2$, then we have $lct(X, C) = \frac{3}{4}$. For the case of a cuspidal rational curve, $lct(X, C) = \frac{5}{6}$. Thus, our statement immediately follows from proposition 3.2. Q.E.D.

Remark 3.4. *Let S be a nonsingular del Pezzo surface of degree d. Then, we have the maximum number r such that $K_S + rC$ is lc for any $C \in |-K_S|$. It is easy to show that such r is $\frac{1}{3}$ (resp. $\frac{1}{2}$) if $d = 9$, 7, or $d = 8$ and Fano index 1 (resp. $d = 5$, 6 or $d = 8$ and Fano index 2).*

Remark 3.5. *If S be a nonsingular del Pezzo surface of degree 1, then $|-K_S|$ has exactly one base point. We can easily check that any element in $|-K_S|$ is nonsingular at this point.*

4. Proof of main theorem.

In this section, we will use the same notations as in the second section.
Proof of main theorem. Since $-K_X$ and $-K_Y$ are ample over T, Surjectivity condition follows from [P99a]. By the same reason, birational map ϕ cannot be an isomorphism in codimension 1 unless it is biregular (see [C95]).

It is enough to check 1-complement condition and Total lc threshold condition by theorem 2.4. Total lc threshold condition immediately follows from corollary 3.3. If $2 \leq d \leq 4$, then it is clear that 1-complement condition holds. In the case of degree 1, 1-complement condition can be derived from remark 3.5. Q.E.D.

Corollary 4.1. *Let X be a del Pezzo fibration over T of degree ≤ 4 with nonsingular scheme-theoretic special fiber. Then, the birational automorphism group of X/T is the same as the biregular automorphism group of X/T.*

Proof. Note that we always assume that birational map is identical on generic fiber. The statement immediately follows from the main theorem. Q.E.D.

As an easy application of theorem 2.4, we can get the following well-known example.

Example 4.2. Let Z be a $\mathbb{P}^1$-bundle over T. Suppose that the special fiber S_Z has no k-rational point. In particular, the residue field k is not algebraically closed. Then, there is no birational transform of Z into another $\mathbb{P}^1$-bundle over T, because the special fiber S_Z satisfies Total lc condition. If S_Z has a k-rational point, then Total lc condition fails. Moreover, it can be birationally transformed into another $\mathbb{P}^1$-bundle over T by elementary transformations.

5. Examples.

If we allow some mild singularities on del Pezzo fibrations, then we can find birational maps of del Pezzo fibrations over T with reduced and irreducible special fiber. In each example, note that one of two del Pezzo fibrations has terminal singularities. Before taking examples, we will state easy lemma which helps us to understand our examples.

Lemma 5.1. *Let $f(x_1, \cdots, x_m, y_1, \cdots, y_n) = g(x_1, \cdots, x_m) + h(y_1, \cdots, y_n)$ be a holomorphic function near $0 \in \mathbb{C}^{m+n}$ and let $D_f = (f = 0)$ on $\mathbb{C}^{m+n}$, $D_g = (g = 0)$ on $\mathbb{C}^m$, and $D_h = (h = 0)$ on $\mathbb{C}^n$. Then*

$$lct(\mathbb{C}^{m+n}, D_f) = min\{1, lct(\mathbb{C}^m, D_g) + lct(\mathbb{C}^n, D_h)\}.$$

Proof. See [Ku99]. Q.E.D.

Example 5.2. This example comes from [C96] and [K97]. Let X and Y be subschemes of $\mathbb{P}^3_{\mathcal{O}}$ defined by equations $x^3 + y^3 + z^2w + w^3 = 0$ and $x^3 + y^3 + z^2w + t^{6n}w^3 = 0$, respectively, where n is a positive integer. Note that X is nonsingular and Y has single singular point of type cD_4 at $p = [0, 0, 0, 1]$. Then, we have a birational map ρ_n of X into Y defined by $\rho_n([x, y, z, w]) = [t^{2n}x, t^{2n}y, t^{3n}z, w]$. Now, we consider a divisor $D \in | - K_X|$ defined by $z = w$. This divisor D has a sort of good divisor because $K_X + S_X + D$ is lc and $D|_{S_X}$ is a nonsingular elliptic curve on S_X. But, the birational transform $\rho_{n*}(D)$ of D by ρ_n is worse than before. First, $\rho_{n*}(D)|_{S_Y}$ is three lines intersecting each other at single point (Eckardt point) transversally on S_Y. Furthermore, we can see that $\rho_{n*}(D)$ on Y is defined by $z = t^{3n}w$. And, the log canonical threshold of $\rho_{n*}(D)$ is $\frac{4n+1}{6n}$ by lemma 5.1, and hence $K_Y + \rho_{n*}(D)$ cannot be lc.

Example 5.3. Let Z and W be subschemes of $\mathbb{P}^3_{\mathcal{O}}$ defined by equations $x^3 + y^2z + z^2w + t^{12m}w^3 = 0$ and $x^3 + y^2z + z^2w + w^3 = 0$, respectively, where m is a positive integer. Here, Z has a singular point of type cE_6 at $[0, 0, 0, 1]$ and W is nonsingular. We have a birational map ψ_m of Z into W defined by $\psi_m([x, y, z, w]) = [t^{2m}x, t^{3m}y, z, t^{6m}w]$. Again, we consider a divisor $H \in | - K_Z|$ defined by $z = w$. For the same reason as above, H is a good divisor. But, the log canonical threshold of the birational transform $\psi_{m*}(H)$ of H by ψ_m is $\frac{5m+1}{6m}$. Therefore, if $m > 1$, then $K_W + \psi_{m*}(H)$ cannot be lc. Note that $\psi_{m*}(H)|_{S_W}$ is a cuspidal rational curve on S_W.

Example 5.4. We consider birational map $\varphi_m = \psi_m^{-1}$ from W to Z, where W, Z and ψ_m are the same as in example 5.3. And, we pay attention to nonsingular divisor $L \in | - K_W|$ on W defined by $x = 0$. Then, we can see that $\varphi_{m*}(L)|_{S_Z}$ consists of a line and a conic intersecting tangentially each other. And, the log canonical threshold of $\varphi_{m*}(L)$ is $\frac{9m+1}{12m}$, and hence $K_Z + \varphi_{m*}(L)$ is not lc.

The following two examples were constructed by M. Grinenko. One is a del Pezzo fibration of degree 2, and the other is of degree 1.

Example 5.5. Let X and Y be subschemes of $\mathbb{P}^3_{\mathcal{O}}(1,1,1,2)$ defined by equations $w^2 + x^3y + x^2yz + z^4 + t^4xy^3 = 0$ and $w^2 + x^3y + xy^3 + x^2yz + t^2z^4 = 0$, respectively, where w is of weight 2. The map $\phi : X \dashrightarrow Y$ defined by $\phi(x,y,z,w) = (x, t^2y, z, tw)$ is birational. Subscheme X has a singular point of type cD_5 at $[0,1,0,0]$. Subscheme Y has two singular points of types cD_6 and cA_1 at $[0,0,1,0]$ and $[1,0,-1,0]$, respectively.

Example 5.6. Let Z be a subscheme of $\mathbb{P}^3_{\mathcal{O}}(1,1,2,3)$ defined by equation $w^2 + z^3 + xy^5 + t^4x^5y = 0$, where z and w are of weight 2 and 3, respectively. Then, we have a birational automorphism α of Z defined by $\alpha(x,y,z,w) = (y, t^2x, t^2z, t^3w)$. Note that Z has a singular point of type cE_8 at $[1,0,0,0]$.

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